

Heat flux in the presence of a gravitational field in a simple dilute fluid: an approach based in general relativistic kinetic theory to first order in the gradients.

D. Brun-Battistini¹, A. Sandoval-Villalazo¹, A. L. Garcia-Perciante²

February 15, 2016

¹Depto. de Fisica y Matematicas, Universidad Iberoamericana Ciudad de Mexico, Prolongacion Paseo de la Reforma 880, Mexico D. F. 01219, Mexico.

²Depto. de Matematicas Aplicadas y Sistemas, Universidad Autonoma Metropolitana-Cuajimalpa, Prol. Vasco de Quiroga 4871, Mexico D.F 05348, Mexico.

Abstract

Richard C. Tolman analyzed the relation between a temperature gradient and a gravitational field in an equilibrium situation. In 2012, Tolman's law was generalized to a non-equilibrium situation for a simple dilute relativistic fluid. The result in that scenario, obtained by introducing the gravitational force through the molecular acceleration, couples the heat flux with the metric coefficients and the gradients of the state variables. In the present paper it is shown, by "suppressing" the molecular acceleration in Boltzmann's equation, that a gravitational field drives a heat flux. This procedure corresponds to the description of particle motion through geodesics, in which a Newtonian limit to the Schwarzschild metric is assumed. The effect vanishes in the non-relativistic regime, as evidenced by the direct evaluation of the corresponding limit.

1 Introduction

The problem of calculating the heat flux in a simple dilute relativistic fluid due to a gravitational field, can be approached from different perspectives. In 1930 Richard C. Tolman considered such system in an equilibrium situation and showed that a gravitational field can balance a temperature gradient, leading to a vanishing heat flux; this is known as *Tolman's law* [1]. Several decades

later, in 2012, an expression for the heat flux in the presence of a linearized gravitational field, was established in a non-equilibrium situation and Tolman's law was recovered when heat flux vanishes and the equilibrium limit is attained [2].

On the other hand, in Ref. [3] the heat flux is calculated in a Schwarzschild isotropic metric in the context of general relativistic kinetic theory concluding that the contribution of the gravitational field vanishes. In that work it is argued that the effect obtained in Ref. [2] may arise from a metric factor not considered in the equilibrium distribution function.

In the present paper it is shown that the gravitational contribution to the heat flux does survive since the absolute molecular acceleration term should not be included in Boltzmann's equation for structureless particles. Moreover, the thermodynamic forces corresponding to the gravitational field are associated to the covariant derivatives present in the formalism. These important conceptual, as well as mathematical, features improve the formalism presented in Ref. [2].

It is relevant to notice that another approach to this problem was presented in 1984 by Wodarzik [4]. In that work the heat flux is expressed using the corresponding Eckart constitutive equation [5], and the well known hydrodynamic generic instabilities that arise if this type of coupling is assumed are obtained [6]. On the other hand, in the present paper we introduce the formal kinetic theory definition of heat flux in the Navier-Stokes regime with no acceleration coupling, instead particle number density, temperature and gravitational potential gradients are present in the final expression, according to the assumptions of linear irreversible thermodynamics. Moreover, Wodarzik assumes that the *fluid bulk* moves following geodesics, argument that can be questioned as individual molecules follow geodesics [7], but stresses deviate the bulk from this type of dynamics. It is well known that the hydrodynamic velocity (\mathcal{U}^ν) satisfies Euler equation [8, 9]:

$$\tilde{\rho} \dot{\mathcal{U}}^\nu = -h^{\nu\alpha} p_{,\alpha}, \quad (1)$$

where $\tilde{\rho}$ is $(\frac{n\varepsilon}{c^2} + p/c^2)$, with n the particle number density, ε the internal energy, p the local pressure and c the speed of light; $h^{\nu\alpha}$ is the spatial projector defined as $g^{\nu\alpha} + \frac{\mathcal{U}^\nu \mathcal{U}^\alpha}{c^2}$, where $p_{,\alpha}$ is the local pressure gradient and $g_{\mu\nu}$ is the metric tensor.

In order to establish the heat flux following the approach described above, the rest of this paper is divided as follows: in Section II we review a few concepts regarding the Boltzmann equation in the general relativistic regime. In Section III, the term $\xi f^{(1)}$ is obtained, following the general relativistic thermodynamic formalism in the BGK approximation within Chapman-Enskog's expansion. The calculation of the heat flux is shown in Section IV and Section V is devoted to final remarks.

2 Basic formalism: relativistic fluids

2.1 Basic elements of general relativity and the Schwarzschild metric

A simple, non-degenerate gas is considered in a Riemannian space where the line element (arc length) is generally expressed as [7, 10]:

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu, \quad (2)$$

where $g_{\mu\nu}$, as defined in Section I, is the metric tensor which is essential when measuring distances in curved space-time. Different metrics may be obtained from Einstein's field equations, depending on the phenomenon to be analyzed. Here, a Schwarzschild metric is used, with the property of being spherically symmetric and static. A signature (1, 1, 1, -1) is considered so that $\mathcal{U}^\nu \mathcal{U}_\nu = -c^2$. In such metric, the line element is given by:

$$ds^2 = \frac{1}{\left(1 - \frac{2GM}{c^2 \tilde{r}}\right)} d\tilde{r}^2 + \tilde{r}^2 (d\theta^2 + \sin^2 \theta d\varphi^2) - \left(1 - \frac{2GM}{c^2 \tilde{r}}\right) (dx^4)^2, \quad (3)$$

with G being the gravitational constant, c the speed of light, M the total mass, source of the gravitational field, and r, θ, φ the spherical coordinates. As in Ref. [3], an isotropic Schwarzschild metric is used [11], for which the substitution $\tilde{r} = r \left(1 + \frac{GM}{2c^2 r}\right)$ is introduced in Eq.(3). Considering $\Phi(r) = \frac{GM}{r}$, the gravitational potential, the Newtonian limit of the Schwarzschild metric, which corresponds to weak field approximation ($\frac{\Phi}{c^2} \ll 1$) reads:

$$\tilde{\eta}_{\mu\nu} = \begin{pmatrix} \left(1 + \frac{2\Phi}{c^2}\right) & 0 & 0 & 0 \\ 0 & \left(1 + \frac{2\Phi}{c^2}\right) & 0 & 0 \\ 0 & 0 & \left(1 + \frac{2\Phi}{c^2}\right) & 0 \\ 0 & 0 & 0 & -\left(1 - \frac{2\Phi}{c^2}\right) \end{pmatrix}. \quad (4)$$

In order to perform the relevant calculations in the next sections it is convenient to recall that in general relativity the velocity is the derivative of the position four-vector with respect to the arc length, so that the molecule's four-velocity is expressed as:

$$v^\mu = c \frac{dx^\mu}{ds} = \frac{cdx^\mu}{c\sqrt{g_{44}}d\tau} = \frac{1}{\sqrt{g_{44}}} \frac{dx^\mu}{d\tau}. \quad (5)$$

Also, in the next section use of Christoffel symbols is made, with the usual definition being:

$$\Gamma_{\alpha\beta}^\mu = \frac{g^{\mu\nu}}{2} \left(\frac{\partial g_{\alpha\nu}}{\partial x^\beta} + \frac{\partial g_{\beta\nu}}{\partial x^\alpha} - \frac{\partial g_{\alpha\beta}}{\partial x^\nu} \right) \quad (6)$$

It must be noticed that the only non-vanishing terms for this metric are the following:

$$\Gamma_{11}^1 = \Gamma_{21}^2 = \Gamma_{31}^3 = \Gamma_{12}^2 = \Gamma_{13}^3 = \frac{\Phi'}{c^2 \left(1 + \frac{2\Phi}{c^2}\right)}, \quad (7)$$

$$\Gamma_{22}^1 = \Gamma_{33}^1 = \Gamma_{44}^1 = -\frac{\Phi'}{c^2 \left(1 + \frac{2\Phi}{c^2}\right)}, \quad (8)$$

$$\Gamma_{14}^4 = \Gamma_{41}^4 = \frac{\Phi'}{c^2 \left(-1 + \frac{2\Phi}{c^2}\right)}. \quad (9)$$

2.2 Boltzmann's general relativistic equation

Boltzmann's equation describes the evolution of the single-particle distribution function and is expressed in special relativity as follows [8, 9]:

$$\dot{f} = J(ff'), \quad (10)$$

where $J(ff')$ is the collisional kernel and, in the special relativistic regime, a dot denotes the total proper time (τ) derivative. Since f is a function of the position (x^μ) and velocity (v^μ) four-vectors, \dot{f} may be written as:

$$\dot{f} = \frac{\partial f}{\partial x^\mu} \dot{x}^\mu + \frac{\partial f}{\partial v^\mu} \dot{v}^\mu. \quad (11)$$

On the other hand, in a general relativistic scenario, \dot{f} corresponds to the arc length derivative ($\dot{f} = \frac{df}{ds}$), with s being the arc length [7]. Also in this approach, we have $\dot{x}^\mu = \frac{dx^\mu}{ds}$ and $\dot{v}^\mu = v^\alpha v_{;\alpha}^\mu = v^\alpha \left(\frac{\partial v^\mu}{\partial x^\alpha} + \Gamma_{\alpha\beta}^\mu v^\beta \right)$. In this paper particles are assumed to lack structure (they don't rotate and they don't have internal degrees of freedom), so that, as a consequence of the field equations, the acceleration $\frac{dv^\mu}{ds}$ is zero, and the molecules move following geodesics, i. e., the following equation is satisfied:

$$\frac{d^2 x^\mu}{ds^2} + \Gamma_{\alpha\beta}^\mu \frac{dx^\alpha}{ds} \frac{dx^\beta}{ds} = 0. \quad (12)$$

The Boltzmann's equation can thus be written as:

$$v^\mu f_{,\mu} = \frac{\partial f}{\partial x^\mu} v^\mu = J(ff'). \quad (13)$$

The collisional kernel can be modeled in a simple form using the BGK approximation [12], so that Eq. (13) becomes:

$$v^\mu f_{,\mu} = -\frac{f - f^{(0)}}{\tau_c}, \quad (14)$$

where τ_c is a relaxation time and $f^{(0)}$ the local equilibrium distribution function.

In order to calculate the heat flux, the first order in the gradients correction to the distribution function ($f^{(1)}$) must be obtained. Following the kinetic theory approach, Chapman-Enskog's method is now used, such that a solution

given by $f = f^{(0)} + f^{(1)}$ is assumed, keeping only the first order in the gradients terms. By substituting this solution in Eq.(14) and after simple algebraic manipulation, one obtains:

$$f^{(1)} = -\tau_c \frac{\partial f^{(0)}}{\partial x^\mu} v^\mu. \quad (15)$$

The next step consists in introducing the functional hypothesis [13] by means of which the factor $\frac{\partial f^{(0)}}{\partial x^\mu}$ is written in terms of the state variables n (the local particle number density), T (the local temperature) and \mathcal{U}^μ (the hydrodynamic fluid velocity):

$$\frac{\partial f^{(0)}}{\partial x^\mu} = \frac{\partial f^{(0)}}{\partial n} \frac{\partial n}{\partial x^\mu} + \frac{\partial f^{(0)}}{\partial T} \frac{\partial T}{\partial x^\mu} + \frac{\partial f^{(0)}}{\partial \mathcal{U}^\alpha} \mathcal{U}_{;\mu}^\alpha. \quad (16)$$

The covariant derivative $\mathcal{U}_{;\mu}^\alpha$ introduced in Eq. (16) is imperative to preserve its invariance. It is interesting to notice that in the definition of the covariant derivative:

$$\mathcal{U}_{;\mu}^\alpha = \frac{\partial \mathcal{U}^\alpha}{\partial x^\mu} + \Gamma_{\mu\nu}^\alpha \mathcal{U}^\nu, \quad (17)$$

the second term will become a thermodynamic force.

The Jüttner (Maxwell-Boltzmann relativistic) function must be considered in order to establish the derivatives in Eq. (16):

$$f^{(0)} = \frac{n}{4\pi m^2 c k_B T K_2(1/z)} e^{\mathcal{U}^\alpha v_\alpha / k_B T}, \quad (18)$$

where $K_\ell(1/z)$ the modified Bessel function of the second kind of order ℓ and $z = k_B T / mc^2$ is the relativistic parameter, with k_B the Boltzmann constant. In order to be precise, the distribution function should include a metric factor $\left(\frac{1}{\sqrt{g^{44}}}\right)$ in the exponential argument, as remarked in Ref.[3], associated with the molecular velocity. However, as v_α and x^α are mutual independent variables, the derivatives with respect to x^α will not include that metric factor. Moreover, the invariant element in the phase space will also have a metric factor \sqrt{g} , with g the metric determinant, but this *will not affect the main result of this paper*, as it will be a general common factor in the resulting integral expression.

Substituting in Eq. (16) the covariant derivative's definition (Eq. 17) and the $f^{(0)}$ partial derivatives that are found in [8], one obtains:

$$\frac{\partial f^{(0)}}{\partial x^\mu} = \frac{f^{(0)}}{n} \frac{\partial n}{\partial x^\mu} + \left(1 - \frac{\gamma_{(k)}}{z} - \frac{\mathcal{G}(1/z)}{z}\right) \frac{\partial T}{\partial x^\mu} + \frac{v_\alpha f^{(0)}}{z c^2} \left(\frac{\partial \mathcal{U}^\alpha}{\partial x^\mu} + \Gamma_{\mu\nu}^\alpha \mathcal{U}^\nu\right), \quad (19)$$

where $\mathcal{G}(1/z) = \frac{K_3(1/z)}{K_2(1/z)}$.

3 The field contribution to $f^{(1)}$

The rest of the calculations in this paper, focus on the field contribution to the heat flux, using the third term on the right hand side term of Eq. (19). Thus, we define

$$f_{\mathcal{U}}^{(1)} = -\tau_c \frac{v_\alpha f^{(0)}}{zc^2} \left(\frac{\partial \mathcal{U}^\alpha}{\partial x^\mu} + \Gamma_{\mu\nu}^\alpha \mathcal{U}^\nu \right) v^\mu. \quad (20)$$

In order to establish the thermodynamic flux, space and time components are separated as follows (latin indexes runs up to 3 and greek ones up to 4):

$$f_{\mathcal{U}}^{(1)} = -\frac{\tau_c f^{(0)}}{zc^2} \left(v_\alpha v^\ell \frac{\partial \mathcal{U}^\alpha}{\partial x^\ell} + v_\alpha v^4 \frac{\partial \mathcal{U}^\alpha}{\partial x^{(4)}} + v_\alpha v^\mu \Gamma_{\mu\nu}^\alpha \mathcal{U}^\nu \right). \quad (21)$$

Notice that Eq. (1) can be expressed as:

$$\dot{\mathcal{U}}^\nu = \mathcal{U}^\mu \mathcal{U}_{;\mu}^\nu = -\frac{h^{\mu\nu} p_{,\mu}}{\tilde{\rho}} \quad (22)$$

and using the covariant derivative definition (Eq. 17):

$$\dot{\mathcal{U}}^\nu = \mathcal{U}^\mu \left(\frac{\partial \mathcal{U}^\nu}{\partial x^\mu} + \Gamma_{\mu\beta}^\nu \mathcal{U}^\beta \right) = -\frac{h^{\mu\nu} p_{,\mu}}{\tilde{\rho}}. \quad (23)$$

The separation of spatial and temporal components yields:

$$\dot{\mathcal{U}}^\nu = \left(\mathcal{U}^\ell \frac{\partial \mathcal{U}^\nu}{\partial x^\ell} + \mathcal{U}^{(4)} \frac{\partial \mathcal{U}^\nu}{\partial x^{(4)}} + \Gamma_{\mu\beta}^\nu \mathcal{U}^\beta \mathcal{U}^\mu \right) = -\frac{h^{\mu\nu} p_{,\mu}}{\tilde{\rho}}, \quad (24)$$

so that the second term in parenthesis turns out to be:

$$\mathcal{U}^{(4)} \frac{\partial \mathcal{U}^\nu}{\partial x^{(4)}} = -\frac{h^{\nu\alpha} p_{,\alpha}}{\tilde{\rho}} - \Gamma_{\mu\beta}^\nu \mathcal{U}^\beta \mathcal{U}^\mu - \mathcal{U}^\ell \frac{\partial \mathcal{U}^\nu}{\partial x^\ell}. \quad (25)$$

The substitution of Eq. (25) in Eq. (21) will allow us to write the time derivatives in terms of first order spatial gradients via the Euler equations. This is necessary to guarantee the existence of the Chapman-Enskog solution [14]. Following such procedure one obtains:

$$\begin{aligned} f_{\mathcal{U}}^{(1)} = & -\frac{\tau_c f^{(0)}}{zc^2} \left(v_\alpha v^\ell \frac{\partial \mathcal{U}^\alpha}{\partial x^\ell} + \frac{v_\alpha v^{(4)}}{\mathcal{U}^{(4)}} \left(-\frac{h^{\nu\alpha} p_{,\alpha}}{\tilde{\rho}} - \Gamma_{\mu\beta}^\nu \mathcal{U}^\beta \mathcal{U}^\mu - \mathcal{U}^\ell \frac{\partial \mathcal{U}^\nu}{\partial x^\ell} \right) \right) \\ & - \frac{\tau_c f^{(0)}}{zc^2} v_\alpha v^\mu \Gamma_{\mu\nu}^\alpha \mathcal{U}^\nu. \end{aligned} \quad (26)$$

In what follows, only the terms depending on Christoffel symbols will be taken into account since they contain the curvature, and thus gravitational effects:

$$f_{[g]}^{(1)} = -\frac{\tau_c f^{(0)}}{zc^2} \left(\frac{-v_\alpha v^{(4)}}{\mathcal{U}^{(4)}} \Gamma_{\beta\lambda}^\alpha \mathcal{U}^\lambda \mathcal{U}^\beta + v_\alpha v^\mu \Gamma_{\mu\nu}^\alpha \mathcal{U}^\nu \right), \quad (27)$$

and calculations will be performed in the comoving frame where $\mathcal{U}^\nu = (0, 0, 0, c)$, $k^\nu \equiv (k^1, k^2, k^3, c)$ and $v_\eta = \gamma_{(k)} k_\eta$ with k_η representing the chaotic velocity (the molecule's velocity measured in the comoving frame) [8, 17] and $\gamma_{(k)} = \left(1 - \frac{k^2}{c^2}\right)^{-1/2}$. With these substitutions we obtain the following expression for the gravitational contribution to the first order in the gradients distribution function:

$$f_{[g]}^{(1)} = -\frac{\tau_c \gamma_{(k)}^2 f^{(0)}}{zc} (k_\alpha k^\mu \Gamma_{\mu 4}^\alpha - c k_\alpha \Gamma_{44}^\alpha). \quad (28)$$

In Ref. [3], the terms corresponding to the ones in brackets in Eq. (28) cancel out as the molecule acceleration term is expressed in terms of a force induced by the field, i. e., $\frac{dv^\mu}{d\tau} = -\Gamma_{\alpha\beta}^\mu v^\alpha v^\beta$, together with the missing covariant derivative of the hydrodynamic velocity in Eq. (18) of Ref. [3].

Before proceeding to the heat flux calculation it is useful to notice that the summation $k_\alpha k^\mu \Gamma_{\mu 4}^\alpha$ vanishes, so that:

$$f_{[g]}^{(1)} = \frac{\tau_c \gamma_{(k)}^2 f^{(0)}}{z} (k_\alpha \Gamma_{44}^\alpha). \quad (29)$$

Equation (29) is the basis of the heat flux field contribution that will be calculated in the next section.

4 Heat flux calculation with a spherically symmetric static metric

Kinetic theory's definition of heat flux in the comoving frame is expressed as follows [15]:

$$J_{[Q]}^\ell = mc^2 \int k^\ell f^{(1)} \gamma_{(k)} (\gamma_{(k)} - 1) d^*K, \quad (30)$$

where the volume element is $d^*K = 4\pi c^3 (\gamma_{(k)}^2 - 1)^{1/2} d\gamma$ [16, 17]. Now, using the expression (29) for $f_{[g]}^{(1)}$, the gravitational field contribution to the hat flux can be written as:

$$J_{[Qg]}^\ell = \frac{\tau_c mc}{z} \int k^\ell f^{(0)} \gamma_{(k)}^3 (\gamma_{(k)} - 1) (c k_\alpha \Gamma_{44}^\alpha) d^*K \quad (31)$$

Performing the calculations (see the appendix for details) it is obtained that:

$$J_{[Qg]}^\ell = \frac{\tau_c m n c^4}{3z} \tilde{\eta}^{\ell\ell} \Gamma_{44}^\ell [1 + 5z \mathcal{G}(1/z) - \mathcal{G}(1/z)] \quad (32)$$

In a Schwarzschild metric, after substitution of the Christoffel symbols, it is found that the gravitational contribution to the heat flux in a simple dilute general relativistic fluid is:

$$J_{[Qg]}^\ell = nm\tau_c c^4 [1 + 5z\mathcal{G}(1/z) - \mathcal{G}(1/z)] \frac{\Phi_{,\ell}}{c^2}, \quad (33)$$

This effect, that is not present in the non-relativistic case, does not vanish in the relativistic regime, neither in special nor in general relativity, at least in the case of a static and symmetrical metric. When the limit when $z \rightarrow 0$ (non-relativistic regime) is considered one obtains:

$$J_{[Qg]}^\ell = \frac{n\tau_c k_B T}{z} \Phi_{,\ell}. \quad (34)$$

This result is in total agreement with the one presented in Refs. [1, 2].

5 Final remarks

It was shown that a gravitational field has a contribution to the heat flux of a simple dilute general relativistic fluid. A static and symmetric metric was assumed during the calculation. Such result is a consequence of using the covariant derivative in Boltzmann's equation for structureless particles that follow geodesic trajectories. The expression obtained leads to the non-relativistic result of the heat flux being coupled solely to the temperature gradient in the corresponding limit, as expected. In the case of flat spacetime and in cartesian coordinates, the covariant derivative vanishes (Christoffel symbols are zero) and the field coupling with the heat flux survives only if a linearized gravity approach is taken into account [2].

In contrast with related works [3], the gravitational field and the particle number density are both coupled with heat flux and will contribute to the entropy production; in other words, the curvature of spacetime itself contributes to the heat flux and produces entropy. A complete evaluation of entropy production ($\sigma = -\frac{k_B}{c} \int J(ff')\varphi d^*v$, with $\varphi = \frac{sf^{(1)}}{f^{(0)}}$) will be addressed in a separate work.

The next step corresponds to the study of tensor effects, which involve viscosity coefficients and the use of Christoffel symbols as thermodynamic forces. In the longer term a similar formalism will be presented to thoroughly analyze the entropy production associated with the field itself.

Acknowledgments

The authors wish to thank MSc. Alma Sagaceta-Mejia and Dr. Humberto Mondragon-Suarez for their valuable comments to this work and acknowledge support from CONACyT through grant number CB2011/167563.

Appendix

In this appendix, details of the gravitational field contribution to the heat flux presented in Section IV are described. Performing the summation over α in Eq. (31) we have:

$$J_{[Qg]}^\ell = \frac{\tau_c m c^2}{z} \int k^\ell f^{(0)} \gamma_{(k)}^3 (\gamma_{(k)} - 1) (k_1 \Gamma_{44}^1 + k_2 \Gamma_{44}^2 + k_3 \Gamma_{44}^3 + k_4 \Gamma_{44}^4) d^* K, \quad (35)$$

which leads to :

$$J_{[Qg]}^\ell = \frac{\tau_c m c^2}{z} \int f^{(0)} \gamma_{(k)}^3 (\gamma_{(k)} - 1) \Gamma d^* K. \quad (36)$$

where factor Γ is:

$$\Gamma = \begin{pmatrix} \Gamma_{44}^1 k^{(1)} k_{(1)} + \Gamma_{44}^2 k^{(1)} k_{(2)} + \Gamma_{44}^3 k^{(1)} k_{(3)} + \Gamma_{44}^4 k^{(1)} k_{(4)} \\ \Gamma_{44}^1 k^{(2)} k_{(1)} + \Gamma_{44}^2 k^{(2)} k_{(2)} + \Gamma_{44}^3 k^{(2)} k_{(3)} + \Gamma_{44}^4 k^{(2)} k_{(4)} \\ \Gamma_{44}^1 k^{(3)} k_{(1)} + \Gamma_{44}^2 k^{(3)} k_{(2)} + \Gamma_{44}^3 k^{(3)} k_{(3)} + \Gamma_{44}^4 k^{(3)} k_{(4)} \end{pmatrix}. \quad (37)$$

In order to write Eq.(36) in covariant form, it will be used that $\tilde{\eta}^{\ell\ell} k_{(\ell)} = k^{(\ell)}$ for $\ell = 1, 2, 3$, and thus $k^{(1)} k_{(1)} = (k^{(1)})^2$. Taking into account that all the terms with the factor $k^{(\mu)} k_{(\nu)}$ for $\mu \neq \nu$ vanish for parity reasons, we shall have:

$$J_{[Qg]}^\ell = \frac{\tau_c m c^2}{z} \int f^{(0)} \gamma_{(k)}^3 (\gamma_{(k)} - 1) \begin{pmatrix} \Gamma_{44}^1 \tilde{\eta}^{11} [k^{(1)}]^2 \\ \Gamma_{44}^2 \tilde{\eta}^{22} [k^{(2)}]^2 \\ \Gamma_{44}^3 \tilde{\eta}^{33} [k^{(3)}]^2 \end{pmatrix} d^* K. \quad (38)$$

Since the three integrals are equal and $[k^{(1)}]^2 + [k^{(2)}]^2 + [k^{(3)}]^2 = k^2$, Eq. (38) can be written as:

$$J_{[Qg]}^\ell = \frac{\tau_c m c^2}{3z} \int f^{(0)} \gamma_{(k)}^3 (\gamma_{(k)} - 1) k^2 \tilde{\eta}^{\ell\ell} \Gamma_{44}^\ell d^* K. \quad (39)$$

The next step is to express the integral in terms of $\gamma_{(k)}$:

$$J_{[Qg]}^\ell = \frac{\tau_c m c^2}{3z} \int_1^\infty f^{(0)} \gamma_{(k)}^3 (\gamma_{(k)} - 1) \frac{c^2 (\gamma_{(k)}^2 - 1)}{\gamma_{(k)}^2} \tilde{\eta}^{\ell\ell} \Gamma_{44}^\ell 4\pi c^3 (\gamma_{(k)}^2 - 1)^{1/2} d\gamma_{(k)}, \quad (40)$$

and after Juttner's distribution function is substituted we obtain that:

$$J_{[Qg]}^\ell = \frac{\tau_c m n c^4}{3z K_2(1/z)} \tilde{\eta}^{\ell\ell} \Gamma_{44}^\ell \int_1^\infty e^{-\gamma_{(k)}/z} \gamma_{(k)} (\gamma_{(k)} - 1) (\gamma_{(k)}^2 - 1)^{3/2} d\gamma_{(k)}, \quad (41)$$

so that:

$$J_{[Qg]}^\ell = \tau_c n k_B T \{1 + 5z \mathcal{G}(1/z) - \mathcal{G}(1/z)\} \Phi^{\ell}, \quad (42)$$

from where the field contribution to the heat flux is given by Eq.(32):

$$J_{[Qg]}^\ell = \frac{\tau_c n m c^2}{z} [1 + 5z\mathcal{G}(1/z) - \mathcal{G}(1/z)] \Phi^{\cdot\ell}.$$

Here use has been made of $z = \frac{k_B T}{mc^2}$, the relativistic parameter and the substitution of $\tilde{\eta}^{\ell\ell}\Gamma_{44}^\ell$ values has been performed.

References

- [1] Tolman, R. C.; Physics. Rev. 35, 904-924 (1930); see also Phys. Rev. **36**, 1791-1798 (1930).
- [2] Sandoval-Villalbaz, A., Garcia-Perciante, A. L. and Brun-Battistini, D.; Phys. Rev. D **86**, 084015 (2012).
- [3] Kremer, G. M.; Relativistic gas in a Schwarzschild metric; J. Stat. Mech. P04016 (2013).
- [4] Wodarzik, U. F.; Phys. Rev. A **30**, 3 (1984).
- [5] Eckart, C.; Phys. Rev. **58**, 267 (1940); Phys. Rev. **58**, 919 (1940).
- [6] Hiscock, A. and Lindblom, L.; Phys. Rev. D **31**, 725 (1985).
- [7] Weinberg, S.; *Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity*; John Wiley & Sons, N. Y. (1972).
- [8] Cercignani, C. and Kremer, G. M.; *The relativistic Boltzmann equation: theory and applications* (Birkhauser Verlag, Basel, 2002).
- [9] Chapman, S. and Cowling, T. G. ; *The mathematical theory of non-uniform gases* (Cambridge Mathematical Library, United Kingdom, 1971), 3rd. ed.
- [10] Kremer, G. M.; The Boltzmann equation in special and general relativity; 28th International Symposium on Rarefied Gas Dynamics (Zaragoza, Spain, July 9-13, 2012).
- [11] See for example Crothers, S. J.; Progress in Physics, (2006) and Buchdahl, H.; International Journal of Theoretical Physics, **24**, 7 (1985).
- [12] Bhatnagar, P. L., Gross, E. P. & Krook, M.; Phys. Rev. **94**, 3 (1954).
- [13] de Groot, S. R., van Leeuwen, W. A. and van der Wert, Ch.; *Relativistic Kinetic Theory* (North Holland Publ. Co., Amsterdam, 1980).
- [14] Courant, R. & Hilbert, D.; Methods of mathematical physics; John Wiley and Sons, vol. 1, New York (1989).
- [15] Garcia-Perciante, A. L., Sandoval-Villalbaz, A. y Garcia-Colin, L. S.; Journal of Non-Equilibrium Thermodynamics, Volume 37, Issue 1, Pages 43-61 (2012).

- [16] Liboff, R. L.; Kinetic Theory: Classical, Quantum and Relativistic Descriptions (Springer, New York, 2003), 3rd ed.
- [17] Garcia-Perciante, A. L. and Mendez, A. R.; Gen. Relativ. Gravit. **43**, 2257 (2011).